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$\pi\pi$-$K\bar{K}$ Model of the ρ Meson*

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The parameters of the ρ meson are calculated in the coupled $\pi\pi$-$K\bar{K}$ problem, using a simple version of the strip approximation. Experimental values are taken for the π, K, K^*, and ϕ masses, while all the coupling constants are assumed to be related through $SU(3)$. The width of the ρ turns out to be 360 MeV, while the mass is 765 MeV. Possible corrections to the model are discussed.

I. INTRODUCTION

IN a previous paper,[1] hereafter referred to as S, a simple version of the strip approximation[2] was applied to the $\pi\pi$ problem. Although this is often (but not always) equivalent to an N/D calculation with a cutoff, it is nevertheless useful to think of it as a strip approximation. This is because the cutoff can be correlated with the strip width, a quantity whose order of magnitude can be estimated by physical arguments.

In the present paper, we shall apply the same approach to the coupled $\pi\pi-K\bar{K}$ problem. The exchanges considered are shown in Fig. 1. Experimental values are taken for the masses of the π, K, K^*, and ϕ mesons, while the ρ mass and $\rho\pi\pi$ coupling are determined self-consistently. The other couplings are assumed to be given in terms of the $\rho\pi\pi$ constant through $SU(3)$. This assumption can be justified *a posteriori* in the case of ρKK coupling without going outside the model.

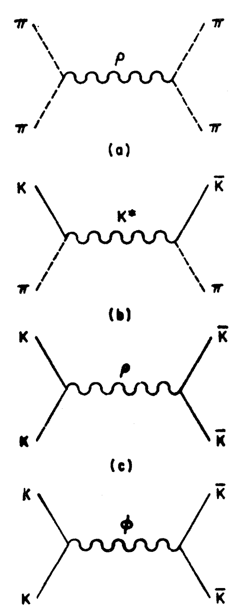

FIG. 1. Possible exchanges in the $\pi\pi-K\bar{K}$ problem.

To get an idea of what we might expect in such a calculation, suppose we consider the degenerate $SU(3)$ case, in which all the members of the PS (pseudoscalar) meson octet have the same mass. This problem can be diagonalized exactly to give a one-channel problem.[3] In this language, one just thinks of PS-PS scattering with V (vector) meson exchange. By a curious accident, it turns out that the crossing matrix element β connecting the V state in the crossed channel to the V state in the direct channel is exactly the same as in the $\pi\pi$ problem, where $\beta=\frac{1}{2}$.

The $\pi\pi$ problem with ρ exchange was considered in S. We shall repeat the calculation for convenience. This time, however, we shall take the strip concept more literally, and assume the strip width to be fixed by the point at which the inelastic dsf (double spectral functions) have their threshold.[2] From S, the partial-wave amplitude is given by

$$B_l(\nu)=\frac{1}{\pi}\int_0^{\nu_1}d\nu'\left(\frac{\nu'^{2l+1}}{\nu'+1}\right)^{1/2}\frac{|B_l(\nu')|^2}{\nu'-\nu}+f_l(\nu)\,, \quad (1)$$

where $B_l(\nu)=[(\nu+1)/\nu^{2l+1}]^{1/2}e^{i\delta}\sin\delta$, $\delta=$phase shift, $\nu=$square of barycentric momentum, and ν_1 is the width of the strip. We shall take ν_1 to correspond to the energy at which the inelastic dsf given by Fig. 2(b) begins. In other words, if ν_R is the position of the V,

$$\nu_1=4\nu_R+3\,. \quad (2)$$

The function $f_l(\nu)$ in our case is the contribution of the V in the crossed channel and in the narrow-width approximation is given by

$$f_l(\nu)=(3\beta/\nu^{l+1})\nu_R\gamma_1(1+2(\nu+1)/\nu_R)Q_l(1+2(\nu_R+1)/\nu)\,. \quad (3)$$

Here, β is the crossing matrix element connecting the V state in the crossed channel to the direct channel state, and γ_1 is the reduced width of the V, i.e., $\nu_R\gamma_1$ is the half-width in the ν variable.

If we solve Eq. (1) by the N/D method, we get

$$B_l(\nu)=N_l(\nu)/D_l(\nu) \quad (4)$$

with

$$N_l(\nu)=f_l(\nu)D_l(\nu)+\frac{1}{\pi}\int_0^{\nu_1}d\nu'\left(\frac{\nu'^{2l+1}}{\nu'+1}\right)^{1/2}\frac{f_l(\nu')N_l(\nu')}{\nu'-\nu} \quad (5)$$

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[1] L. A. P. Balázs, Phys. Rev. 134, B1315 (1964).
[2] G. F. Chew and S. C. Frautschi, Phys. Rev. 123, 1478 (1961).
[3] R. H. Capps, Phys. Rev. Letters 10, 312 (1963).

and

$$D_l(\nu)=1-\frac{1}{\pi}\int_0^{\nu_1}d\nu'\left(\frac{\nu'^{2l+1}}{\nu'+1}\right)^{1/2}\frac{N_l(\nu')}{\nu'-\nu}. \qquad (6)$$

Now it was shown in S that we can make a threshold approximation for $f_l(\nu)$. Then

$$N_l(\nu)=f_l(\nu)=b_l=3\beta\gamma_1\frac{\nu_R+2}{(\nu_R+1)^{l+1}}\frac{\pi^{1/2}\Gamma(l+1)}{2^{l+1}\Gamma(l+\frac{3}{2})}. \qquad (7)$$

If we also make a linear approximation for Eq. (6) with value and slope given correctly at the physical threshold, we obtain

$$D_l(\nu)=1-b_l(h_l+\nu h_{l-1}), \qquad (8)$$

where

$$\pi h_l=\int_0^{\nu_1}d\nu'\left(\frac{\nu'^{2l+1}}{\nu'+1}\right)^{1/2}. \qquad (9)$$

In the V state, Eqs. (7) and (8) give

$$\gamma_1=-N_1(\nu_R)/D_1'(\nu_R)=h_0^{-1}, \qquad (10)$$

which enables us to find b_1 in terms of ν_R through Eq. (7). If this b_1 is substituted into Eq. (8), then the condition for the V resonance is

$$D_1(\nu_R)=0=1-\frac{\beta(\nu_R+2)}{(\nu_R+1)^2}\left[\frac{h_1}{h_0}+\nu_R\right] \qquad (11)$$

with

$$h_1=\frac{1}{\pi}\{[\nu_1(\nu_1+1)]^{1/2}-\ln[(\nu_1)^{1/2}+(\nu_1+1)^{1/2}]\}$$

and

$$h_0=\frac{2}{\pi}\ln[(\nu_1)^{1/2}+(\nu_1+1)^{1/2}].$$

Now, if ν_1 is assumed to be known, Eq. (11) is just a quadratic equation in ν_R. If we assume that ν_1 is itself given in terms of ν_R through Eq. (2), the equation for ν_R is somewhat more complicated. All we have to do, however, is plot Eq. (11) and see where it vanishes. This gives $\nu_R=0.7$. Equation (10) then yields $\gamma_1=0.98$.

In the $SU(2)$ model ($\pi\pi$ scattering), these parameters correspond to the ρ having a mass of 360 MeV and a half-width of 50 MeV. Since the masses of the PS octet are badly broken, it is not clear how the corresponding $SU(3)$ model should be interpreted. Perhaps the most relevant quantity is the reduced width of the decay of the ρ into two pions, which is given by $\gamma_{\rho\pi\pi}=\frac{2}{3}\gamma_1=0.65$. The corresponding experimental value, corresponding to a full width of 100 MeV, is $\gamma_{\rho\pi\pi}=0.16$ if the mass is taken to be 765 MeV.

II. THE $\pi\pi$-$K\bar{K}$ SYSTEM IN THE 1^- STATE

To get a result which can be unambiguously compared with experiment, we must do a calculation in which the π and K masses have their correct values. We then use the 1^- amplitudes

$$T_{ij}=s^{1/2}(u_{ij}-\delta_{ij})/(4iq_i^{3/2}q_j^{3/2}), \qquad (12)$$

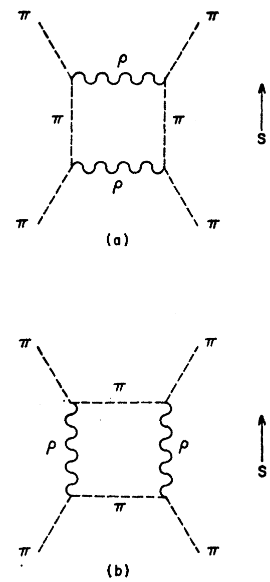

FIG. 2. Box diagrams constituting nearby dsf regions. In each case, it is understood that the diagram is accompanied by the corresponding crossed graph.

where

$s^{1/2}$ = total barycentric energy,
u = unitary 2×2 matrix,
$q_1=[(s/4)-1]^{1/2}$ = momentum of pion (mass $m_1=1$),
$q_2=[(s/4)-m^2]^{1/2}$ = momentum of kaon (mass $m_2=m$).

By a straightforward generalization of the procedure followed in S, we can write

$$T(s)=N(s)D^{-1}(s), \qquad (13)$$

where N and D satisfy the equations

$$N(s)=f(s)D(s)+\frac{1}{\pi}\int_4^{s_1}ds'\rho(s')\frac{f(s')N(s')}{s'-s}, \qquad (14)$$

$$D(s)=1-\frac{1}{\pi}\int_4^{s_1}ds'\rho(s')\frac{N(s')}{s'-s}, \qquad (15)$$

with

$$\rho_{ij}(s)=(2q_i^3/s^{1/2})\delta_{ij}\theta(q_i^2). \qquad (16)$$

As in the one-channel case, we shall take the strip width to be given by the threshold of Fig. 2(b). In other words, $s_1=4m_\rho^2$, where m_ρ is the mass of the ρ. The function $f(s)$ comes from the exchanges of Fig. 1. The π, K, K^*, and ϕ masses are now assumed to be all different and are taken from experiment.[4] The ρ mass is left undetermined and will be found self-consistently, as will also the reduced width γ_{11} of the ρ into two pions in the s variable. The remaining reduced widths will be determined from γ_{11} through $SU(3)$.[3] If we also approximate the elements of $f(s)$ by their threshold values, we

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[4] W. H. Barkas and A. H. Rosenfeld, University of California Radiation Laboratory Report UCRL Report 8030 (unpublished).

obtain

$$f_{11}(s)=f_{11}(4)=((m_\rho^2+4)/2m_\rho^4)\gamma_{11}\,, \tag{17}$$

$$f_{22}(s)=f_{22}(4m^2)=\left[\frac{3}{2}\frac{m_\phi^2+4m^2}{4m_\phi^4}-\frac{1}{2}\frac{m_\rho^2+4m^2}{4m_\rho^4}\right]\gamma_{11}\,, \tag{18}$$

$$f_{12}(s)=f_{21}(s)=f_{12}(4m^2)=(\sqrt{2}\gamma_{11}/4m_{K^*}{}^2 \times(m_{K^*}{}^2+m^2-1)^2)\{[m_{K^*}{}^2-(m+1)^2] \times[m_{K^*}{}^2-(m-1)^2]+8m^2m_{K^*}{}^2\}\,, \tag{19}$$

where m_ϕ and m_{K^*} are the ϕ and K^* masses, respectively. It can be shown that the expressions (17) to (19) are reasonable approximations to the contributions of Fig. 1.

Since D is normalized to unity at infinity and since the threshold approximation for f corresponds to a pole at infinity, we have $f(s)=N(s)$. Thus Eq. (15) becomes

$$D_{ij}(s)=\delta_{ij}-\alpha_i(s)f_{ij}\,, \tag{20}$$

where

$$\alpha_i(s)=\frac{1}{\pi}\int_4^{s_1}ds'\frac{\rho_{ii}(s')}{s'-s}\,. \tag{21}$$

If, as in the one-channel case, we make a linear approximation for $D_{ij}(s)$ by expanding $\alpha_i(s)$ about $s=4m_i^2$, we obtain

$$\alpha_i(s)=(m_i^2/\pi)\{[c_i(c_i+1)]^{1/2} -3\ln[(c_i)^{1/2}+(c_i+1)^{1/2}]\} +(s/2\pi)\ln[(c_i)^{1/2}+(c_i+1)^{1/2}]\,, \tag{22}$$

where $c_i=(s_1/4m_i^2)-1$. The position of the ρ is now given by

$$\det D(m_\rho^2)=0\,, \tag{23}$$

while the residue matrix is

$$\gamma=-[Nd]\Big/\left[\frac{d}{ds}\det D\right]_{s=m_\rho^2}\,, \tag{24}$$

where d is the cofactor of D. In particular, the reduced width of the ρ into two pions is

$$\gamma_{11}=[f_{11}-(f_{11}f_{22}-f_{12}f_{21})\alpha_2(m_\rho^2)]\Big/\left[\frac{d}{ds}\det D\right]_{s=m_\rho^2}\,. \tag{25}$$

Now Eqs. (23) and (24) are both quadratic equations in γ_{11}. If we eliminate the γ_{11}^2 term, we obtain at $s=m_\rho^2$

$$\gamma_{11}=\frac{g_{11}-[(\alpha_1'\alpha_2+\alpha_1\alpha_2')/\alpha_1\alpha_2]}{(g_{11}g_{22}-g_{12}g_{21})\alpha_2+(g_{11}\alpha_1'+g_{22}\alpha_2')-(g_{11}\alpha_1+g_{22}\alpha_2)[(\alpha_1'\alpha_2+\alpha_1\alpha_2')/\alpha_1\alpha_2]}\,, \tag{26}$$

where $g_{ij}=\gamma_{11}^{-1}f_{ij}$.

Since the g_{ij} are independent of γ_{11}, Eq. (26) determines γ_{11} for any assumed m_ρ^2. This enables us to calculate $\det D(m_\rho^2)$ and check whether it vanishes. By trying various m_ρ^2, it was found that Eq. (23) is satisfied for $m_\rho^2=30$. The corresponding value of γ_{11} is 2.9. If we plot the $\pi\pi$ 1^- cross section

$$\sigma_1=\frac{12\pi\gamma_{11}^2[q_1^4/(q_1^2+1)]}{(s-m_\rho^2)^2+\gamma_{11}^2[q_1^6/(q_1^2+1)]}\,, \tag{27}$$

we obtain a half-width for the distribution of 180 MeV.

It is not possible to check whether our $SU(3)$ model is reasonable for the $K^*K\pi$ and $KK\phi$ couplings without going beyond our scheme. However, we can check the ρKK coupling by calculating it through Eq. (24), which gives $\gamma_{22}=1.36$ for the reduced width. The corresponding result calculated in terms of the $\rho\pi\pi$ vertex through $SU(3)^3$ is $\gamma_{22}=(\gamma_{11}/2)=1.45$, a result which is not too different from the calculated value. This seems to suggest that using $SU(3)$ to obtain relations between reduced widths may be quite a reasonable procedure, even if the masses are badly broken. It is also an interesting fact that the reduced width γ_{11} calculated in the degenerate calculation of the preceding section is 2.6 in the s variable, a result which is about the same as the value 2.9 obtained in the nondegenerate calculation of this section. Thus it might be not unreasonable to use even unbroken $SU(3)$ symmetry to calculate reduced widths.

III. POSSIBLE CORRECTIONS TO THE MODEL

We shall now discuss possible corrections to the model. The most obvious one, which is still within the strip approximation, would be the effect of higher waves. These presumably come mainly from the contributions of the type represented by Fig. 2 to the strip regions. To get an idea of their effect, it should be sufficient to evaluate the s-channel absorptive part $A_s{}^I$ of Fig. 2(a) at various values of s and t in the region $s<s_1$, $t<s_1$ and compare it with the corresponding contribution of 1^- waves. Of course, in evaluating Fig. 2(a), we have to subtract out its contribution to the 1^- state which is already explicitly taken into account for $s<s_1$. In the zero-width approximation for the ρ, we thus have in the $I=1$, $\pi\pi$ state

$$A_s{}^1(s,t)=\frac{9\gamma_{11}^2(m_\rho^2-4)}{4[s(s-4)]^{1/2}}\left(1+\frac{2s}{m_\rho^2-4}\right)^2 \times\left\{2\frac{\tan^{-1}\{[t/(t_B-t)]^{1/2}\}}{[t/(t_B-t)]^{1/2}} -\frac{\ln t_B-2\ln[(-u)^{1/2}-(t_B-u)^{1/2}]}{[u(t_B-u)]^{1/2}} -\frac{6}{s-4}\left(1+\frac{2t}{s-4}\right)\left(\left[J_0(s)+\frac{2}{s-4}J_1(s)\right]\ln t_B -2\left[I_0(s)+\frac{2}{s-4}I_1(s)\right]\right)\right\}\,, \tag{28}$$

where

$$t_B=4m_\rho^2\left(1+\frac{m_\rho^2}{s-4}\right),$$

$$J_0(s)=\ln t_B-2\ln[(t_B+s-4)^{1/2}-(s-4)^{1/2}],$$

$$J_1(s)=\tfrac{1}{2}t_BJ_0(s)-[(s-4)(t_B+s-4)]^{1/2},$$

$$I_0(s)=\tfrac{1}{4}J_0(s)\{\ln t_B+2\ln[(t_B+s-4)^{1/2}-(s-4)^{1/2}]\},$$

$$I_1(s)=\tfrac{1}{2}t_BI_0(s)-\tfrac{1}{2}(s-4)-[(s-4)(t_B+s-4)]^{1/2}\times\ln[(s-4)^{1/2}-(t_B+s-4)^{1/2}].$$

Actually, if we are only interested in estimating the effect of Eq. (28) at moderate energies, it should be sufficient to keep only the lowest term in an expansion of $A_s^1(s,t)$ in powers of (t/t_B). This gives

$$A_s^1(s,t)=\frac{\gamma_{11}^2(m_\rho^2-4)}{[s(s-4)]^{1/2}}\left(1+\frac{2s}{m_\rho^2-4}\right)^2\times\left(\frac{s-4}{t_B}\right)^3P_3\left(1+\frac{2t}{s-4}\right)\frac{36t_B-64(s-4)}{175t_B^2}. \quad (29)$$

Equation (29) is to be compared with the corresponding P-wave contribution to the absorptive part, which has the form

$$C_s^1(s,t)=\frac{\sigma_1}{16\pi}[s(s-4)]^{1/2}P_1\left(1+\frac{2t}{s-4}\right) \quad (30)$$

with σ given by Eq. (27). The results for Eqs. (29) and (30) are shown for various values of s and t in Table I. We see that the effect of Fig. 2 is small.

To go beyond the strip approximation, we must have a systematic procedure for including all possible effects, inelastic as well as elastic. One such procedure, for which the lowest order approximation is essentially Eq. (1), is the multiple-impulsive peripheral expansion suggested by Cutkosky.[5] In this expansion, diagrams are computed just as they would be with ordinary Feynman rules except that continuous as well as discrete masses make up the internal lines. In higher order diagrams, one also has to be careful not to double-count effects which are already included in lower order diagrams. The lowest order approximation is given by diagrams typified by Fig. 3, in which all legs could be continuum states and the vertices essentially consist of a finite number of low partial-wave amplitudes. The main difficulty with the approach is that higher order diagrams usually diverge and so one has to introduce regulators. These correspond to a cutoff at an energy which presumably can be identified with the strip width.

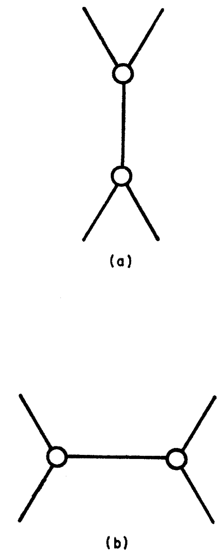

FIG. 3. Typical lowest order diagrams. In degenerate PS-PS scattering, these just give Eq. (1) in a particular partial wave.

TABLE I. The absorptive parts $A_s^1(s,t)$ and $C_s^1(s,t)$ at various values of s and t.

s	t	$A_s^1(s,t)$	$C_s^1(s,t)$
30	0	5.1×10^{-4}	3.2
30	30	5.0×10^{-2}	10.7
60	0	1.8×10^{-2}	2.3
60	30	7.6×10^{-1}	4.7

In addition to bringing in higher effects, the above method can be used to improve the lowest order approximation. This is because a given set of diagrams can be used to calculate the partial-wave amplitudes, which themselves constitute the vertices in the diagrams. Thus we have a set of integral equations for these partial-wave amplitudes. For instance, as we increase the number of diagrams in degenerate PS-PS scattering, we have to include their effect in $f_l(\nu)$ in Eq. (1) at the same time. This should give an improved lowest order term when we solve the resulting Eq. (1).

Now because the inclusion of more diagrams presumably treats higher energy effects more and more accurately, the cutoff has to be increased each time, since it is just a device for damping effects which are too large because they were not treated properly. The simplest way of doing this is to take as the cutoff the normal threshold of the lowest diagram which is left out. Another method would be to set all strip widths equal and fix them by the requirement that the calculated mass of the pion be equal to the assumed value. This does not introduce any parameters since one mass has to be arbitrary to fix the mass scale.

If we allow the number of diagrams to tend to infinity, the strip width should tend to infinity at the same time, and the sum of all the diagrams should approach the correct answer. It is not clear, of course, whether this procedure converges. Perhaps it converges only by increasing at the same time the number of partial waves in the basic vertices. However, the fact that at least the low-energy absorptive part of the next highest diagram in $\pi\pi$ scattering (Fig. 2) is small compared with the lowest order term (as we saw at the beginning of this section) suggests that the possibility of such a convergence is not entirely unreasonable.

[5] R. E. Cutkosky, Nucl. Phys. 37, 57 (1962).